# A General Form for Functions of Binomial Type

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## Introduction

The sum of two variables raised to a positive integer exponent obeys the binomial expansion:

$$(x+y)^k = \sum_{n=0}^{k} \binom{k}{n} x^n y^{(k-n)}$$

Functions of binomial type are functions $Q_k(x)$ defined over integers $k \geq 0$ and real $x$ that obey the analogous binomial expansion:

$Q_k(x+y) = \sum_{n=0}^{k} \binom{k}{n} Q_n(x) Q_{k-n}(y)$, and $Q_0(x) = 1$.

In this paper, $Q_k(x)$ will be used to refer to a function of $x$, with functions of binomial type being sequences of functions $Q_k(x)$ with $k$ acting as an index for those functions.

Notice that:

$$Q_k(x+y) = Q_k(x) + Q_k(y) + \sum_{n=1}^{k-1} \binom{k}{n} Q_n(x) Q_{k-n}(y)$$

The summation only contains $Q$s with positive subscripts lower than $k$, with no reference to $Q_k$. This leaves the coefficient of $x$ in $Q_k(x)$ not fixed, because $c(x+y) = cx + cy$, so any $cx$ may be added to $Q_k(x)$, and the function will remain of binomial type from $Q_1$ through $Q_k$.

Assuming that $Q_k(x)$ is continuous in $x$, the linear term $cx$ is the only term that is not fixed in $Q_k(x)$. All other terms are determined by the functions $Q_1(x)$ through $Q_{k-1}(x)$. As such, every function of binomial type can be expressed in terms of an infinite sequence that in this paper will be referred to as $a$, starting at $a_1$, with each element $a_k$ being the coefficient of $x$ in $Q_k(x)$. To determine the correct general form it will help to first assume that $Q_k(x)$ is a polynomial (which later we will prove must be the case). Now letting $y = x$ in the definition we get $Q_k(2x) = \sum_{n=0}^{k} \binom{k}{n} Q_n(x) Q_{k-n}(x)$ and if we plug $Q_1(x) = \sum_{i=0}^{\infty} c_i x^i$ into that we can match up the coefficients of $x$ to show that all the $c_i$ must be zero except $c_1$. Repeating this process we get the general form for the first 5 functions of binomial type as:

$Q_1(x) = a_1 x,$
$Q_2(x) = a_2 x + a_1^2 x^2,$
$Q_3(x) = a_3 x + 3a_2 a_1 x^2 + a_1^3 x^3,$
$Q_4(x) = a_4 x + 4a_3 a_1 x^2 + 3a_2^2 x^2 + 6a_2 a_1^2 x^3 + a_1^4 x^4,$
$Q_5(x) = a_5 x + 5a_4 a_1 x^2 + 10a_3 a_2 x^2 + 10a_3 a_1^2 x^3 + 15a_2^2 a_1 x^3 + 10a_2 a_1^3 x^4 + a_1^5 x^5.$

Notice that in every term, the subscripts of $a$ add the the subscript of $Q$, and the exponent of $x$ is the number of $a$-variables multiplied together in the term (the constant portions of the terms have a combinatorial explanation that will be explained later). For instance, $Q_5(x)$ contains the term $10a_3 a_1 a_1 x^3$. $3 + 1 + 1 = 5$, and $\{3, 1, 1\}$ contains 3 terms. Sequences of integers that add to $n$ are known as partitions of $n$. Every $Q_n$ contains one term for each partition of $n$. Now for some background on partitions:

A partition is an unordered collection of positive integers, and may contain repeated elements. In this paper, the lowercase letters $r$, $s$, and $t$ will be used to represent partitions. A subscript will be used to access the elements of a partition, as in: $r_i$, but because partitions are unordered, the subscript must only be an iterator, and all elements must be treated symmetrically. For instance, $\sum_i r_i$ refers to the sum of the elements of $r$, but $r_3$ and $\sum_i i r_i$ are meaningless. A magnitude sign will be used to denote the number of elements in a partition. For instance, $|r|$ refers to the number of elements of $r$. $\#r(i)$ refers to the number of terms in $r$ equal to $i$. I'll use an example to help clarify the notation. Consider the partions of 8. Note that there are 22 such partitions, but let's consider $r$ to be the particular one $\{3, 2, 2, 1\}$. Then we have $|r| = 4$ and $\#r(1) = 1, \quad \#r(2) = 2, \textit{ and } \quad \#r(3) = 1.$

With this notation, the general form for functions of binomial type, proved to be correct later in this paper, can be written as:

$$Q_k(x) = \sum_{\substack{r \text{ such that} \\ \sum r_i = k}} \frac{k! \prod_i (a_{r_i} x)}{\prod_i (r_i!) \prod_i \#r(i)!}$$

$\prod(a_{r_i} x)$ is a product of terms in the $a$ sequence with the subscripts corresponding to a partition $r$, times $x^{|r|}$. The coefficient is the number of unique ways of sorting $k$ objects into the partition $r$, without distinguishing between parts of the same size. $\frac{k!}{\prod_i (r_i!)}$ is the multinomial coefficient, which counts the number of ways of sorting $k$ objects into the partition $r$, distinguishing between parts of the same size. $\prod_i \#r(i)!$ is the product of the factorial of the number of terms in $r$ of each value, and dividing by this offsets the fact that the multinomial coefficient distinguishes between parts of the same size. As an example, the term corresponding to the partition $3 + 1 + 1$ is $\frac{5! a_3 a_1^2 x^3}{(3!1!1!)(2!1!)}$, which is indeed a term in $Q_5(x)$.

This polynomial form is already known, and a proof can be found in The Umbral Calculus by Steven Roman. This paper presents an alternative angle on the problem. First I will prove that any function satisfying the polynomial form is of binomial type. I will follow this with two proofs of the converse which states that all functions of binomial type that are continuous in $x$ can be written in the polynomial form for some sequence $a_i$.

Before presenting the proofs, the concept of a subpartition used in this paper must be defined.

A subpartition $s$ of $r$ is defined as a partition such that all elements in $s$ are also included in $r$, and no number appears more often in $s$ than in $r$. (Informally, a subpartition is to a partition as a subset is to a set.) The complement of $s$ within $r$ is defined as the partition $t$ such that, for all $i$, $\#s(i) + \#t(i) = \#r(i)$. Whenever $s$ is explicitly defined as a subpartition of $r$, $t$ will be used to denote the complement of $s$ within $r$.

Consider again the {3,2,2,1} partition of 8 example we used earlier. I used color to distinquish between the two 2's, but this doesn't change the number of partitions of 8 since by definition the order of the parts doesn't matter. (i.e. {3,2,2,1} is the same partition.) Distinguishing the elements will prove useful however when considering the subpartitions since then the number of terms in

$\sum_{\substack{subpartions\\ s\,of\,r}} (\ldots)$ is $2^{|r|}$, because every element in $r$ can be included in $s$ or not. (So for example {3,2} and {3,2} are counted separately). It will also be useful to count equivalent subpartitions only once. For this I will use the notation $\sum_{\substack{subpartions\\ s\,of\,r\,(eq)}} (\ldots)$ where the $'eq'$ indicates that we are including only one from among each set of equivalent subpartions (for example, there is only one copy of {3,2} in the subpartitions of {3,2,2,1} when counting in this manner). This of course will have fewer terms than the general summation if there are repeated elements in the partition. Given a value that appears in $r$ $n$ times, there are $\binom{n}{m}$ different ways to pick from those identical elements to construct a subpartition of $r$ that includes $m$ of them. Therefore the number of times each subpartition $s$ is included in such a summation is:

$$\prod_i \binom{\#r(i)}{\#s(i)} = \frac{\prod_i \#r(i)!}{\prod_i \#s(i)! \prod_i \#t(i)!}$$ which implies that:

$$\sum_{\substack{subpartions\\ s\,of\,r}} (\ldots) = \sum_{\substack{subpartions\\ s\,of\,r\,(eq)}} (\ldots) \frac{\prod_i \#r(i)!}{\prod_i \#s(i)! \prod_i \#t(i)!}$$ or conversely, that:

$$\sum_{\substack{subpartions\\ s\,of\,r\,(eq)}} (\ldots) = \sum_{\substack{subpartions\\ s\,of\,r}} (\ldots) \frac{\prod_i \#s(i)! \prod_i \#t(i)!}{\prod_i \#r(i)!}$$

The proof as well as the converse proof both use this fact.

# Proof

given: $Q_k(x) = \sum_{\substack{r\,such\\that\\ \sum r_i=k}} \frac{k!\prod_i(a_{r_i}x)}{\prod_i(r_i!)\prod_i \#r(i)!}$ prove: $Q_k(x+y) = \sum_{n=0}^{k}\binom{k}{n}Q_n(x)Q_{k-n}(y)$.

We start out as we would if we were just proving the binomial theorem:

$$(x+y)^n = \underbrace{(x+y)\,(x+y)\,(x+y)\,\ldots\,(x+y)}_{n\,of\,these}$$

This product can be expanded by repeated application of the distributive law which will yield $2^n$ terms since for each of the $(x+y)$ factors we must use either the x or the y term. Recasting this in our language of partitions, let's choose $r$ to be the partition $\{1,2,3,\ldots,n\}$ and chose the subpartion $s$ to include the $i^{th}$ element of $r$ if we use the x in the $i^{th}$ factor and not include that element if we use the y. So now we can write:

$$(x+y)^{|r|} = \sum_{\substack{subpartions\\s\,of\,r}} x^{|s|}y^{|t|}$$

Note that although we used a particular partition $r$ the above equation is true for any partition $r$ because of our use of the more general method of summing over subpartitions. From this equation a simple counting step gives us the binomial theorem, however we will take a different path to achieve a more general result. We start by muliplying both sides of the above equation by the product of the $a_i$ sequence:

$$(x+y)^{|r|}\prod_i a_{r_i} = \sum_{\substack{subpartions\\s\,of\,r}} x^{|s|}y^{|t|}\prod_i a_{r_i} = \sum_{\substack{subpartions\\s\,of\,r}} x^{|s|}y^{|t|}\prod_i a_{s_i}\prod_i a_{t_i}$$

$\prod_i a_{r_i}$ iterates $|r|$ times, so this can be rewritten as:

$$\prod_i(a_{r_i}(x+y)) = \sum_{\substack{subpartions\\s\,of\,r}} \prod_i(a_{s_i}x)\prod_i(a_{t_i}y) = \sum_{\substack{subpartions\\s\,of\,r\,(eq)}} \frac{\prod_i(a_{s_i}x)\prod_i(a_{t_i}y)\prod_i \#r(i)!}{\prod_i \#s(i)!\prod_i \#t(i)!}$$

$\prod_i(a_{r_i}(x+y))$ appears in the polynomial form of $Q_k(x+y)$, so I can make the substitution:

$$Q_k(x+y) = \sum_{\substack{r\,such\\that\\ \sum r_i=k}} \frac{k!\prod_i(a_{r_i}(x+y))}{\prod_i(r_i!)\prod_i \#r(i)!} = \sum_{\substack{r\,such\\that\\ \sum r_i=k}} \left\{ \frac{k!}{\prod r_i!} \sum_{\substack{subpartions\\s\,of\,r\,(eq)}} \frac{\prod_i(a_{s_i}x)\prod_i(a_{t_i}y)}{\prod_i \#s(i)!\prod_i \#t(i)!} \right\}$$

$$= k! \sum_{\substack{r\,such\,that\\ \sum r_i=k}} \sum_{\substack{subpartions\\s\,of\,r\,(eq)}} \frac{\prod_i(a_{s_i}x)\prod_i(a_{t_i}y)}{\prod_i(r_i!)\prod_i \#s(i)!\prod_i \#t(i)!}$$

$$= k! \sum_{\substack{r\,such\,that\\ \sum r_i=k}} \sum_{\substack{subpartions\\s\,of\,r\,(eq)}} \frac{\prod_i(a_{s_i}x)}{\prod_i(s_i!)\prod_i \#s(i)!}\cdot\frac{\prod_i(a_{t_i}y)}{\prod_i(t_i!)\prod_i \#t(i)!}$$

Note that this double summation includes every possible pair of partitions $s$ and $t$ such that $\sum_i s_i + \sum_i t_i = k$, each appearing exactly once.

Thus, it can be rewritten:

$$Q(x+y) = k! \sum_{n=0}^{k} \sum_{\substack{r \, such \\ that \\ \sum r_i = n}} \frac{\prod_i (a_{r_i} x)}{\prod_i (r_i!) \prod_i \#r(i)!} \sum_{\substack{r \, such \\ that \\ \sum r_i = k-n}} \frac{\prod_i (a_{r_i} y)}{\prod_i (r_i!) \prod_i \#r(i)!}$$

Now if we substitute $k! = \binom{k}{n} n! \, (k-n)!$ and move it into the sumation we get:

$$Q_k(x+y) = \sum_{n=0}^{k} \binom{k}{n} \sum_{\substack{r \, such \\ that \\ \sum r_i = n}} \frac{n! \prod_i (a_{r_i} x)}{\prod_i (r_i!) \prod_i \#r(i)!} \sum_{\substack{r \, such \\ that \\ \sum r_i = k-n}} \frac{(k-n)! \prod_i (a_{r_i} y)}{\prod_i (r_i!) \prod_i \#r(i)!}$$

$$= \sum_{n=0}^{k} \binom{k}{n} Q_n(x) Q_{k-n}(y)$$

Q.E.D.

# Proof of converse

It has already been shown that any function that satisfies the polynomial form is of binomial type. Assuming that $Q_k(x)$ is continuous, each sequence $a$ must uniquely determine at most one function of binomial type, since every term in $Q_k(x)$ except for the constant term is exactly defined from $Q_1(x)$ through $Q_{k-1}(x)$. Therefore, all functions of binomial type that are continuous in $x$ satisfy the polynomial form.

# Alternate proof of converse

Assume that $Q_k(x)$ is continuous in $x$. If $f(x)$ is continuous and $f(x+y) = f(x)+f(y)$, then $f(x) = cx$. Therefore,

$Q_1(x+y) = Q_1(x) + Q_1(y)$

$Q_1(x) = a_1 x = \frac{1!}{1!\,1!}\, a_1 x$

The polynomial form holds true for $Q_1$. Now using induction we assume:

$$Q_k(x) = \sum_{\substack{r \text{ such that} \\ \sum r_i = k}} \frac{k! \prod_i (a_{r_i} x)}{\left[\prod_i (r_i!)\right] \prod_i \#r(i)!} \quad \text{for all } k < R$$

Let's define: $C_R(x) = Q_R(x) - \sum_{\substack{r \text{ such} \\ \text{that} \\ \sum r_i = R}} \frac{R! \prod_i (a_{r_i} x)}{\left[\prod_i (r_i!)\right] \prod_i \#r(i)!}$ such that $C_R(x)$ contains no term of the form $c \cdot x$, since that would be included in the term $a_R x$ in $Q_R(x)$

We would have our result if we could prove that: $C_R(x) = 0$

In the original equation, substitute: $x \to \frac{x}{2},\ y \to \frac{x}{2}$

$Q_k(x) = \sum_{n=0}^{k} \binom{k}{n} Q_n(\frac{x}{2}) Q_{k-n}(\frac{x}{2})$

$$\binom{R}{S} Q_S(\frac{x}{2}) Q_T(\frac{x}{2}) = \frac{R!}{S!T!} \left\{ \sum_{\substack{s \text{ such} \\ \text{that} \\ \sum s_i = S}} \frac{S! \prod_i (a_{s_i} \frac{x}{2})}{\left[\prod_i (s_i!)\right] \prod_i \#s(i)!} \right\} \bullet \left\{ \sum_{\substack{t \text{ such} \\ \text{that} \\ \sum t_i = T}} \frac{T! \prod_i (a_{t_i} \frac{x}{2})}{\left[\prod_i (t_i!)\right] \prod_i \#t(i)!} \right\}$$

where $0 < S < R$; $T = R - S$

$$\binom{R}{0} Q_0(\frac{x}{2}) Q_R(\frac{x}{2}) = \binom{R}{R} Q_R(\frac{x}{2}) Q_0(\frac{x}{2}) = C_R(\frac{x}{2}) + \sum_{\substack{r \text{ such that} \\ \sum r_i = R}} \frac{R! \prod_i (a_{r_i} \frac{x}{2})}{\left[\prod_i (r_i!)\right] \prod_i \#r(i)!}$$

$$Q_R(x) = \sum_{S=0}^{R} \binom{R}{S} Q_S(\frac{x}{2}) Q_{R-S}(\frac{x}{2})$$

$$= 2C_R(\frac{x}{2}) + R! \sum_{S=0}^{R} \sum_{\substack{s \text{ such} \\ \text{that} \\ \sum s_i = S}} \frac{\prod_i (a_{s_i} \frac{x}{2})}{\left[\prod_i (s_i!)\right] \prod_i \#s(i)!} \sum_{\substack{t \text{ such} \\ \text{that} \\ \sum t_i = R-S}} \frac{\prod_i (a_{t_i} \frac{x}{2})}{\left[\prod_i (t_i!)\right] \prod_i \#t(i)!}$$

$$= 2C_R(\frac{x}{2}) + R! \sum_{\substack{r\, such\\ that\\ \sum r_i = R}} \sum_{\substack{subpartions\\ s\, of\, r\, (eq)}} \frac{\prod_i (a_{s_i} \frac{x}{2})}{\left[\prod_i (s_i!)\right] \prod_i \#s(i)!} \cdot \frac{\prod_i (a_{t_i} \frac{x}{2})}{\left[\prod_i (t_i!)\right] \prod_i \#t(i)!}$$

$$= 2C_R(\frac{x}{2}) + R! \sum_{\substack{r\, such\\ that\\ \sum r_i = R}} \sum_{\substack{subpartions\\ s\, of\, r}} \frac{\prod_i (a_{s_i} \frac{x}{2})}{\left[\prod_i (s_i!)\right] \prod_i \#s(i)!} \cdot \frac{\prod_i (a_{t_i} \frac{x}{2})}{\left[\prod_i (t_i!)\right] \prod_i \#t(i)!} \cdot \frac{\prod_i \#s(i)! \prod_i \#t(i)!}{\prod_i \#r(i)!}$$

$$= 2C_R(\frac{x}{2}) + \sum_{\substack{r\, such\, that\\ \sum r_i = R}} \frac{R! \prod_i (a_{r_i} \frac{x}{2})}{\left[\prod_i (r_i!)\right] \prod_i \#r(i)!} \sum_{\substack{subpartions\\ s\, of\, r}} 1$$

$$= 2C_R(\frac{x}{2}) + \sum_{\substack{r\, such\, that\\ \sum r_i = R}} \frac{2^{|r|} R! \prod_i (a_{r_i} \frac{x}{2})}{\left[\prod_i (r_i!)\right] \prod_i \#r(i)!}$$

$$= 2C_R(\frac{x}{2}) + \sum_{\substack{r\, such\, that\\ \sum r_i = R}} \frac{R! \prod_i (a_{r_i} x)}{\left[\prod_i (r_i!)\right] \prod_i \#r(i)!}$$

and from the definition of $C_R$:

$$Q_R(x) = C_R(x) + \sum_{\substack{r\, such\, that\\ \sum r_i = R}} \frac{R! \prod_i (a_{r_i} x)}{\left[\prod_i (r_i!)\right] \prod_i \#r(i)!}$$

$$C_R(x) = 2C_R\left(\frac{x}{2}\right)$$

The coefficient of $x$ in $Q_R(x)$is not fixed. Since $Q_k(x)$ is continuous, $C_R(x)$ must also be continuous, and thus all polynomials for which $C_R(x) = 2C_R\left(\frac{x}{2}\right)$ are of the form $C_R(x) = cx$, but $a_R$ was defined in such a way that the coefficient of x in $C_R(x)$is 0.

$C_R(x) = 0$

$$\therefore Q_k(x) = \sum_{\substack{r\, such\, that\\ \sum r_i = k}} \frac{k! \prod_i (a_{r_i} x)}{\left[\prod_i (r_i!)\right] \prod_i \#r(i)!}$$

Q.E.D.