Mahonian Triangle

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The Triangle of Mahonian numbers (or Mahonian triangle), referred to here as the function $M(r, c)$, where r and c are integers, is defined by the following generating function:

$$\sum_{c=0}^{\infty} M(r,c)x^c = \prod_{b=0}^{r} \sum_{a=0}^{b} x^a.$$

This definition is equivalent to:
$M(0, 0) = 1$,
$M(0, c) = 0$ for $c \neq 0$,
$M(r,c) = \sum_{k=0}^{r} M(r-1, c-k)$ for $r > 0$.
The first few rows of the Mahonian triangle, starting with $M(0, 0)$, are:

1															
1	1														
1	2	2	1												
1	3	5	6	5	3	1									
1	4	9	15	20	22	20	15	9	4	1					
1	5	14	29	49	71	90	101	101	90	71	49	29	14	5	1

This paper deals with the relationship between the Mahonian triangle and another two-dimensional function referred to as $P(r, c)$ and by the name "Truncated Euler Product triangle". Note that the word "triangle" appearing in the name of both 2D functions is a misnomer, as the number of elements in each row grows quadratically rather than linearly in both of them. The Truncated Euler Product triangle is defined by the following generating function:

$$\sum_{c=0}^{\infty} P(r,c)x^c = \prod_{b=1}^{r+1} (1 - x^b).$$

This definition is equivalent to:
$P(-1, 0) = 1$,
$P(-1, c) = 0$ for $c \neq 0$,
$P(r, c) = P(r-1, c) - P(r-1, c-r-1)$ for $r > -1$.
The first few rows of the Truncated Euler Product triangle, starting with $P(0, 0)$, are:

1	-1																				
1	-1	-1	1																		
1	-1	-1	0	1	1	-1															
1	-1	-1	0	0	2	0	0	-1	-1	1											
1	-1	-1	0	0	1	1	1	-1	-1	-1	0	0	1	1	-1						
1	-1	-1	0	0	1	0	2	0	-1	-1	-1	-1	0	2	0	1	0	0	-1	-1	1

These functions have the following relations, both of which are proved later in this paper:

Equation 1: $M(r,c) = \sum_{k=0}^{c} \binom{r+c-k}{r} P(r,k)$,

Equation 2: $P(r,c) = \sum_{k=0}^{c} (-1)^{c-k} \binom{r+1}{c-k} M(r,k)$.

The first relation is of greater interest because of the utter lack of importance of the Truncated Euler Product triangle for anything other than computing the Mahonian triangle. I now present the proofs.

Proof of Equation 1

Lemma prove: $\left(\frac{1}{1-x}\right)^{r+1} = \sum_{a=0}^{\infty} \binom{r+a}{a} x^a$

$\binom{r+b}{b}$ =Number of distinct ways to line up b stars and r bars.

With the bars marking the dividers between groups, this is equivalent to the number of ways to sort b interchangable stars into $r+1$ distinct groups.

With each group representing an instance of $\sum_{a=0}^{\infty} x^a$, this is equivalent to the number of ways of extracting x^b from $\left(\sum_{a=0}^{\infty} x^a\right)^{r+1}$.

$$\therefore \sum_{a=0}^{\infty} \binom{r+a}{a} x^a = \left(\sum_{a=0}^{\infty} x^a\right)^{r+1} = \left(\frac{1}{1-x}\right)^{r+1}.$$

Theorem prove: $M(r,c) = \sum_{k=0}^{c} \binom{r+c-k}{r} P(r,k)$

$$\sum_{c=0}^{\infty} M(r,c)x^c = \prod_{b=0}^{r} \sum_{a=0}^{b} x^a$$
$$= \prod_{b=0}^{r} \left(\frac{1-x^{b+1}}{1-x}\right)$$
$$= \left(\frac{1}{1-x}\right)^{r+1} \prod_{b=1}^{r+1} (1-x^b)$$
$$= \sum_{a=0}^{\infty} \binom{r+a}{a} x^a \sum_{c=0}^{\infty} P(r,c)x^c$$
$$= \sum_{a=0}^{\infty} \binom{r+a}{a} x^a \sum_{c=a}^{\infty} P(r,c-a)x^{c-a}$$

$P(r,c-a) = 0$ for $c-a<0$, so c can be extended down to 0 without changing the sum.

$$= \sum_{a=0}^{\infty} \sum_{c=0}^{\infty} \binom{r+a}{a} P(r,c-a)x^c$$

switching the order of the summations and replacing $c-a$ with k, we get

$$= \sum_{c=0}^{\infty} \sum_{k=-\infty}^{c} \binom{r+c-k}{c-k} P(r,k)x^c$$
$$= \sum_{a=0}^{\infty} \sum_{k=0}^{c} \binom{r+c-k}{r} P(r,k)x^c$$
$$\therefore M(r,c) = \sum_{k=0}^{c} \binom{r+c-k}{r} P(r,k)$$

Q.E.D.

Proof of Equation 2

$$\prod_{b=1}^{r+1} (1-x^b) = \left(\prod_{b=0}^{r} \sum_{a=0}^{b} x^a\right) (1-x)^{r+1}$$
$$\sum_{c=0}^{\infty} P(r,c)x^c = \left(\sum_{c=0}^{\infty} M(r,c)x^c\right) \left(\sum_{c=0}^{\infty} \binom{r+1}{c} (-x)^c\right)$$
$$\therefore P(r,c) = \sum_{k=0}^{c} (-1)^{c-k} \binom{r+1}{c-k} M(r,k)$$

Q.E.D.

Connecting Proof

Both relations have now been proved correct individually. I now present a simple proof, not relying on the definitions of the functions, that the relations are equivalent to each other.

Lemma prove: $\binom{r+k}{r} = \sum_{\sum s_i = k} (-1)^{|s|+k} \prod_i \binom{r+1}{s_i}$. Here, s is used to represent finite sequences of positive integers. These are like partitions except that the order matters (i.e., $2,1$ is different than $1,2$). s_i represents terms in the sequence, and $|s|$ represents the number of terms in the sequence.

We start with the lemma from the proof of Equation 1.

$$\left(\frac{1}{1-x}\right)^{r+1} = \sum_{k=0}^{\infty}\binom{r+k}{r}x^k$$

$$1 = (1-x)^{r+1}\sum_{k=0}^{\infty}\binom{r+k}{r}x^k$$

$$= \sum_{a=0}^{\infty}(-1)^a\binom{r+1}{a}x^a\sum_{k=0}^{\infty}\binom{r+k}{r}x^k$$

$= \sum_{k=0}^{\infty}\sum_{a=0}^{k}(-1)^a\binom{r+1}{a}\binom{r+k-a}{r}x^k$. a only goes up to k because if $k-a$ goes bellow 0, $\binom{r+k-a}{r}$ is not included in the sum.

$\sum_{a=0}^{k}(-1)^a\binom{r+1}{a}\binom{r+k-a}{r} = 0$ for $k > 0$; $= 1$ for $k = 0$.

$\binom{r+k}{r} = -\sum_{a=1}^{k}(-1)^a\binom{r+1}{a}\binom{r+k-a}{r}$ for $k > 0$; $= 1$ for $k = 0$.

Now, we recurse this equation, making the $\binom{r+k-a}{r}$ feed back in as $\binom{r+k}{r}$, and the a terms become a finite sequence of positive integers that adds to k.

$$\binom{r+k}{r} = \sum_{\sum s_i = k}(-1)^{|s|+k}\prod_i\binom{r+1}{s_i}.$$

Theorem given: $P(r,c) = \sum_{k=0}^{c}(-1)^{c-k}\binom{r+1}{c-k}M(r,k)$

prove: $M(r,c) = \sum_{k=0}^{c}\binom{r+c-k}{r}P(r,k)$

$$P(r,c) = M(r,c) + \sum_{k=0}^{c-1}(-1)^{c-k}\binom{r+1}{c-k}M(r,k)$$

$$M(r,c) = P(r,c) - \sum_{k=0}^{c-1}(-1)^{c-k}\binom{r+1}{c-k}M(r,k)$$

We recurse this equation, making $M(r,k)$ feed back in as $M(r,c)$, and get

$$M(r,c) = P(r,c) - \sum_{k_1=0}^{c-1}(-1)^{c-k_1}\binom{r+1}{c-k_1}\bullet$$

$$\left\{P(r,k_1) - \sum_{k_2=0}^{k_1-1}(-1)^{k_1-k_2}\binom{r+1}{k_1-k_2}\left[P(r,k_2) - \sum_{k_3=0}^{k_2-1}(-1)^{k_2-k_3}\binom{r+1}{k_2-k_3}\left(P(r,k_3) - \sum_{k_4=0}^{k_3-1}(-1)^{k_3-k_4}\binom{r+1}{k_3-k_4}\ldots\right)\right]\right\}$$

$$= P(r,c) - \left[\sum_{k_1=0}^{c-1}(-1)^{c-k_1}\binom{r+1}{c-k_1}P(r,k_1)\right] + \left[\sum_{k_1=0}^{c-1}(-1)^{c-k_1}\binom{r+1}{c-k_1}\sum_{k_2=0}^{k_1-1}(-1)^{k_1-k_2}\binom{r+1}{k_1-k_2}P(r,k_2)\right]$$

$$- \left[\sum_{k_1=0}^{c-1}(-1)^{c-k_1}\binom{r+1}{c-k_1}\sum_{k_2=0}^{k_1-1}(-1)^{k_1-k_2}\binom{r+1}{k_1-k_2}\sum_{k_3=0}^{k_2-1}(-1)^{k_2-k_3}\binom{r+1}{k_2-k_3}P(r,k_3)\right]$$

$$+ \left[\sum_{k_1=0}^{c-1}(-1)^{c-k_1}\binom{r+1}{c-k_1}\sum_{k_2=0}^{k_1-1}(-1)^{k_1-k_2}\binom{r+1}{k_1-k_2}\sum_{k_3=0}^{k_2-1}(-1)^{k_2-k_3}\binom{r+1}{k_2-k_3}\sum_{k_4=0}^{k_3-1}(-1)^{k_3-k_4}\binom{r+1}{k_3-k_4}P(r,k_4)\right] - \ldots$$

$$= P(r,c) - \left[\sum_{k_1=0}^{c-1}(-1)^{c-k_1}\binom{r+1}{c-k_1}P(r,k_1)\right] + \left[\sum_{k_1=0}^{c-1}\sum_{k_2=0}^{k_1-1}(-1)^{c-k_2}\binom{r+1}{c-k_1}\binom{r+1}{k_1-k_2}P(r,k_2)\right]$$

$$- \left[\sum_{k_1=0}^{c-1}\sum_{k_2=0}^{k_1-1}\sum_{k_3=0}^{k_2-1}(-1)^{c-k_3}\binom{r+1}{c-k_1}\binom{r+1}{k_1-k_2}\binom{r+1}{k_2-k_3}P(r,k_3)\right]$$

$$+ \left[\sum_{k_1=0}^{c-1}\sum_{k_2=0}^{k_1-1}\sum_{k_3=0}^{k_2-1}\sum_{k_4=0}^{k_3-1}(-1)^{c-k_4}\binom{r+1}{c-k_1}\binom{r+1}{k_1-k_2}\binom{r+1}{k_2-k_3}\binom{r+1}{k_3-k_4}P(r,k_4)\right] - \ldots$$

Now we sort subterms by the value of the inner k_i in relation to c, and get

$$M(r,c) = \sum_{k=0}^{c}\sum_{\sum s_i = k}(-1)^{|s|+k}P(r,c-k)\prod_i\binom{r+1}{s_i}.$$

Multiplying both sides of our lemma by $P(r,c-k)$, we get

$$\sum_{k=0}^{c}\binom{r+k}{r}P(r,c-k)=\sum_{k=0}^{c}\sum_{\sum s_i=k}(-1)^{|s|+k}P(r,c-k)\prod_i\binom{r+1}{s_i}=M(r,c)$$
$$\sum_{k=0}^{c}\binom{r+k}{r}P(r,c-k)=\sum_{k=0}^{c}\binom{c+r-k}{r}P(r,k)$$
$$\therefore M(r,c)=\sum_{k=0}^{c}\binom{r+c-k}{r}P(r,k)$$
Q.E.D.

Naive Formula for M(r,c)

$M(r,c)=\sum_{k=0}^{r}M(r-1,c-k)$ for $r>0$.
Recurse this equation, feeding $M(r-1,c-k)$ back in as the new $M(r,c)$, and you will get
$$M(r,c)=\sum_{k_1=0}^{r}\sum_{k_2=0}^{r-1}\sum_{k_3=0}^{r-2}\sum_{k_4=0}^{r-3}\ldots\sum_{k_{r-1}=0}^{2}\sum_{k_r=0}^{1}M\left(0,c-\sum_{n=1}^{r}k_n\right).$$
Plug in $M(0,c)=\delta(c)$, and
$$M(r,c)=\sum_{k_1=0}^{r}\sum_{k_2=0}^{r-1}\sum_{k_3=0}^{r-2}\sum_{k_4=0}^{r-3}\ldots\sum_{k_{r-1}=0}^{2}\sum_{k_r=0}^{1}\delta\left(c-\sum_{n=1}^{r}k_n\right).$$
In other words, M(r,c) is the number of sequences of nonnegative integers with r elements, and such that the nth element no bigger than r+1-n, that add to c.

Formula for M(r,c) and P(r,c) for c<r+2

Define the function $p(c)$ with the following generating function:
$\sum_{c=0}^{\infty}p(c)x^c=\prod_{b=1}^{\infty}(1-x^b)$. Notice that $p(c)=P(\infty,c)$. Euler's pentagonal number theorem states that $p(k)=(-1)^a$ if there is some integer a such that $\frac{a(3a-1)}{2}=c$; 0 otherwise.
From the recursive formula for $P(r,c)$, we get
$$P(r_2,c)=P(r_1,c)-\sum_{a=r_1}^{r_2-1}P(a,c-a-2)$$
$$p(c)=P(r,c)-\sum_{a=r}^{\infty}P(a,c-a-2)$$
$P(r,c)=0$ for $c<0$, so
$P(r,c)=p(c)$ for $c<r+2$.
Notice that, from Equation 1, this implies that
$M(r,c)=\sum_{k=0}^{c}\binom{r+c-k}{r}p(k)$ for $c<r+2$.

Remark

Without taking into account its relations with the Truncated Euler Product triangle, it is still easy to see that the Mahonian triangle must be of the form
$M(r,c)=\sum_{k=0}^{c}\binom{r+c-k}{r}f(k)$ for $c<r+2$, for some function $f(k)$ (which is proven above to be $p(k)$). Feel free to skip to the next section if you don't care.

Proof First, notice that $M(r,0)=1\binom{r}{r}$, which fits the formula. Now, assuming that
$M(r,c-1)=\sum_{k=0}^{c-1}\binom{r+c-1-k}{r}f(k)$ for $c-1<r+2$, we will prove that
$M(r,c)=\sum_{k=0}^{c}\binom{r+c-k}{r}f(k)$ for $c<r+2$, for some value of $f(c)$.
$M(r,c)=\sum_{k=0}^{r}M(r-1,c-k)$ for $r>0$.

$M(r,c)=0$ for $c<0$.

$M(r,c)=\sum_{k=0}^{c} M(r-1,c-k)=\sum_{k=0}^{c} M(r-1,k)$ for $c<r+1$.

$M(r,c-1)=\sum_{k=0}^{c-1} M(r-1,k)$ for $c<r+1$.

$M(r,c)=M(r-1,c)+M(r,c-1)$ for $c<r+1$.

$M(r,c)=M(c,c)+\sum_{a=0}^{r-c-1} M(r-a,c-1)$ for $c<r+1$.

$M(r-a,c-1)=\sum_{k=0}^{c-1}\binom{(r-a)+(c-1)-k}{(r-a)}f(k)$ for $c-1<r-a+1$

$M(r,c)=M(c,c)+\sum_{k=0}^{c-1}\sum_{a=0}^{r-c-1}\binom{(r-a)+(c-1)-k}{(r-a)}f(k)$

$\sum_{a=r-c}^{r}\sum_{k=0}^{c-1}\binom{(c-1)+(r-a)-k}{(r-a)}f(k)$ and $M(c,c)$ depend only on c (the r's cancel out), so it is valid to define

$f(c)=M(c,c)-\sum_{a=r-c}^{r}\sum_{k=0}^{c-1}\binom{(r-a)+(c-1)-k}{(r-a)}f(k)$

$M(r,c)=f(c)+\sum_{k=0}^{c-1}\sum_{a=0}^{r}\binom{(r-a)+(c-1)-k}{(r-a)}f(k)$

$\sum_{a=0}^{r}\binom{(r-a)+(c-1)-k}{(r-a)}=\sum_{a=0}^{r}\binom{a+c-1-k}{a}=\binom{r+c-k}{r}$

$M(r,c)=f(c)+\sum_{k=0}^{c-1}\binom{r+c-k}{r}f(k)=\sum_{k=0}^{c}\binom{r+c-k}{r}f(k)$. The inductive step is completed. Q.E.D.

General Formula for P(r,c)

$P(r,c)=P(r-1,c)-P(r-1,c-r-1)$ for $r>-1$
$P(r-1,c)=P(r,c)+P(r-1,c-r-1)$
$P(r,c)=P(r+1,c)+P(r,c-r-2)$

$P(r,c)=P(c-1,c)+\sum_{k=0}^{c-r-2} P(r+k,c-r-2-k)=p(c)+\sum_{k=0}^{c-r-2} P(r+k,c-r-2-k)$

Recurse this equation, iteratively feeding $P(r+k,c-r-2-k)$ back in as the new $P(r,c)$, and you will get

$P(r,c)=p(c)+\sum_{k_1=0}^{c-r-2} p(k_1)+\sum_{k_2=0}^{2k_1-c} p(k_2)+\sum_{k_3=0}^{2k_2-k_1} p(k_3)+\sum_{k_4=0}^{2k_3-k_2} p(k_4)+\sum_{k_5=0}^{2k_4-k_3} p(k_5)+\sum_{k_6=0}^{2k_5-k_4} p(k_6)+\sum_{k_7=0}^{2k_6-k_5} p(k_7)+...$

If $c<n(r+2)$, you can stop after k_{n-1}, because the rest of the imbedded sums will all be empty.

Formula for M(r,c) for c<2r+4

Notice that the above formula for $P(r,c)$ implies that

$M(r,c)=\sum_{k_0=0}^{c}(-1)^{c-k_0}\binom{r+c-k_0}{r}\left(p(k_0)+\sum_{k_1=0}^{k_0-r-2} p(k_1)+\sum_{k_2=0}^{2k_1-c} p(k_2)+\sum_{k_3=0}^{2k_2-k_1} p(k_3)+\sum_{k_4=0}^{2k_3-k_2} p(k_4)+\sum_{k_5=0}^{2k_4-k_3} p(k_5)+...\right)$

Most of this is a combinatorial nightmare to simplify, but for $c<2r+4$,

$M(r,c)=\sum_{k_0=0}^{c}(-1)^{c-k_0}\binom{r+c-k_0}{r}\left(p(k_0)+\sum_{k_1=0}^{k_0-r-2} p(k_1)\right)$

$\sum_{k_0=0}^{c}\binom{r+c-k_0}{r}\sum_{k_1=0}^{k_0-r-2} p(k_1)=\sum_{k_0=-r-2}^{c-r-2}\binom{c-k_0-2}{r}\sum_{k_1=0}^{k_0} p(k_1)$

$=\sum_{k_0=0}^{c-r-2}\binom{c-k_0-2}{r}\sum_{k_1=0}^{k_0} p(k_1)$

$=\sum_{k_1=0}^{c-r-2} p(k_1)\sum_{k_0=k_1}^{c-r-2}\binom{c-k_0-2}{r}$

$$= \sum_{k_1=0}^{c-r-2} p(k_1) \sum_{k_0=r}^{c-k_1-2} \binom{k_0}{r}$$

$$= \sum_{k_1=0}^{c-r-2} p(k_1) \binom{c-k_1-1}{r+1}$$

$$= \sum_{k=0}^{c} p(k) \binom{c-k-1}{r+1}$$

$$\therefore M(r,c) = \sum_{k=0}^{c} p(k) \left[\binom{r+c-k}{r} + \binom{c-k-1}{r+1} \right] \text{ for } c < 2r+4$$